The spectrum of a ring

The functor Spec lets us associate a geometric object to each ring.

Def: Let R be a ring. Let $\operatorname{Spec} R$ denote the set of prime ideals of R.

We define a topology, the Zariski topology, on $\operatorname{Spec} R$ as follows:

For any subset $I \subseteq R$, $V(I) := \{P \in \operatorname{Spec} R \mid I \subseteq P\}$.

Note: $V(I) = V(\text{ideal gen. by } I)$, so we can restrict our definition to ideals.

In fact, by the previous section $V(I) = V(\operatorname{rad}(I))$.

$V(I)$ are the closed sets of the Zariski topology.

Note $V(0) = \operatorname{Spec} R$ and $V(R) = \emptyset$.

Claim:

a.) $\bigcap V(I_\lambda) = V\left(\sum_\lambda I_\lambda\right)$

b.) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$

Pf: a.) $P \in \bigcap V(I_\lambda) \iff P \supseteq \bigcup I_\lambda \iff P \in V\left(\sum_\lambda I_\lambda\right)$.

b.) $IJ \subseteq I \cap J$, so $V(IJ) \supseteq V(I \cap J)$.

If $P \in V(I) \cup V(J)$ then $P \supseteq I$ or J so $P \supseteq I \cap J$,
so $V(I) \cup V(J) \subseteq V(I \cap J)$.

Now suppose $P \supseteq IJ$ but $P \not\supseteq I$. i.e. there is
$u \in I$ but $u \notin P$. $\forall$ $v \in J$, $uv \in P$, so $v \in P$.
Thus $J \subseteq P$. $\Rightarrow$ $V(I) \cup V(J) \supseteq V(IJ)$. $\square$

Note that a single point set $\{P\}$ is closed iff P is maximal. The subset of max'l ideals is <u>maxspec(R)</u>.

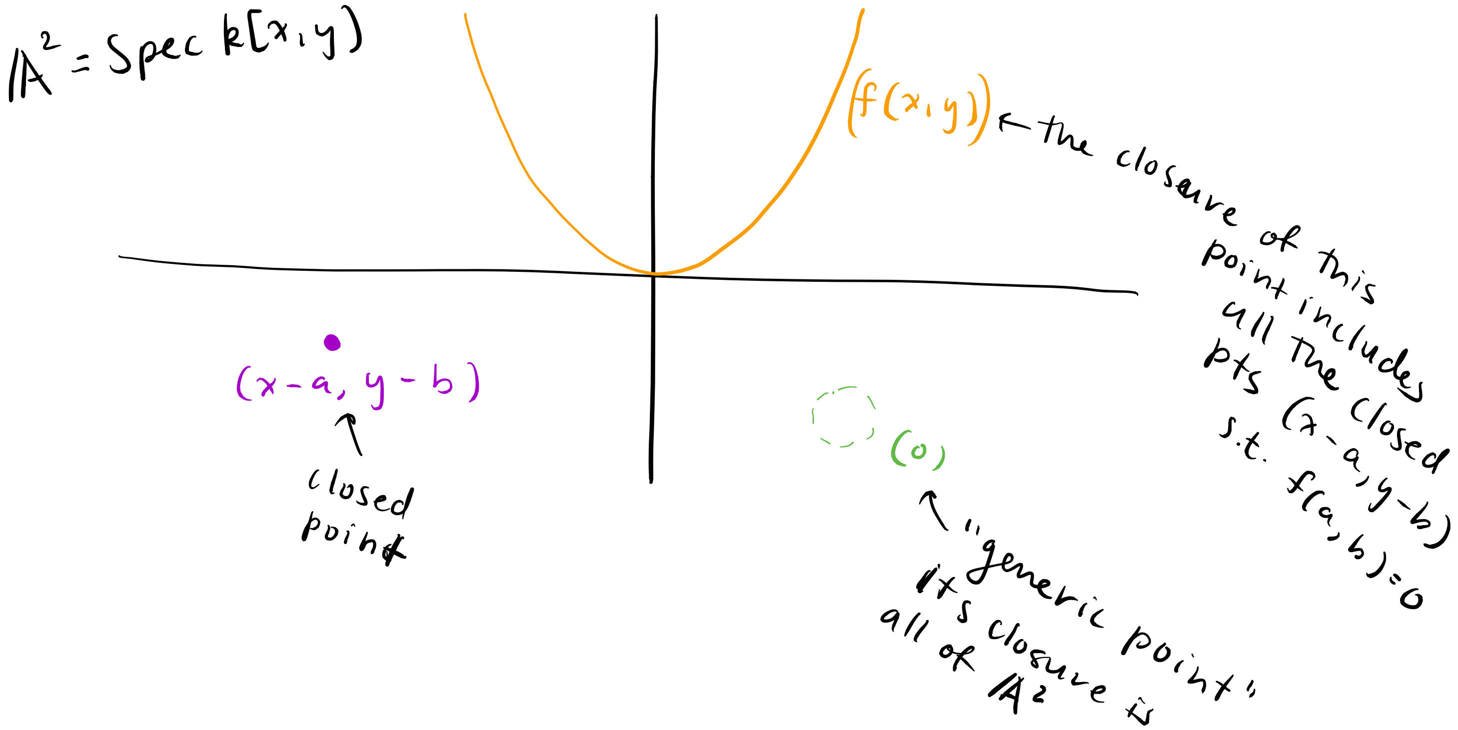
Geometrically, these are the points we usually consider.

Ex: 1.) $\operatorname{Spec} k[x] = \{(f(x))\} = \mathbb{A}^1_k$ f a polynomial.
If k is algebraically closed, these are in one-to-one bijection w/ points on k, and (0).

$\max\operatorname{Spec}(k[x]) =$ points of k.

2.) $\max\operatorname{Spec} k[x,y] = \{(x-a, y-b) \mid a,b \in k\}$ $(k = \bar{k})$
so the closed points are in bijection w/ k^2.

The other points of spec correspond to curves defined by irreducible polynomials $f(x,y)$.



(More generally, $\mathbb{A}^n_k = \text{Spec}\left(k[x_1, \ldots, x_n]\right)$)

Spec as a functor

Spec is a contravariant functor from rings to topological spaces:

If $\varphi: R \to S$ is a ring map, then define

$$\text{Spec}(\varphi): \text{Spec}(S) \to \text{Spec}(R)$$
$$P \longmapsto \varphi^{-1}(P)$$

This induced map is continuous, i.e. the preimage of a closed set is closed:

Claim: If $\varphi: R \to S$ is a ring map, and $I \subseteq R$, then $(\operatorname{Spec} \varphi)^{-1}(V(I)) = V(\varphi(I))$.

Pf: P is in the LHS $\iff \varphi^{-1}(P) \in V(I) \iff I \subseteq \varphi^{-1}(P)$ $\iff \varphi(I) \subseteq P \iff P$ is in the RHS. $\square$

Connection to quotients and localizations

We already know that the prime ideals of R/I and $R[U^{-1}]$ are in correspondence w/ certain subsets of $\operatorname{Spec}(R)$. In fact:

Claim:

a.) The map $\operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ is a homeomorphism* of $\operatorname{Spec}(R/I)$ w/ $V(I) \subseteq \operatorname{Spec}(R)$

b.) The map $\operatorname{Spec}(R[U^{-1}]) \to \operatorname{Spec}(R)$ is a homeomorphism of $\operatorname{Spec}(R[U^{-1}])$ w/ $Y \subseteq \operatorname{Spec}(R)$ where $Y = \{P \mid U \cap P = \emptyset\}$.

Pf: See HW 2.

* a homeomorphism is a continuous, invertible map whose inverse is also continuous.